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THE INSCRIPTION OF REGULAR POLYGONS.

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CHAPTER I.

From the time of the early Greeks, geometers have taken a decided interest in the inscription of regular polygons. However, practically nothing was added to the subject as left us by Euclid for twenty centuries—until Gauss published his immortal researches thereon. Up to Gauss' time geometers felt certain that the only regular polygons that could be inscribed geometrically were those having 2^x, 3.2^x, 5.2^x, and 15.2^x sides.

Gauss proved that every regular polygon having for the number of its sides a prime number of the form 2^x+1, or the product of any number of different primes of that form, or a power of 2 times such a number, is geometrically inscriptible. Thus he added to Euclid's list those of 17, 34, 51, 257, etc., sides.

Further, he gave a method of deducing the simplest equations upon whose solution depends the inscription of any particular regular polygon. He made the inscription of the regular n-gon depend upon the solution of the binomial equation $x^n-1=0$, *i. e.*, upon finding the nth roots of unity. When n is odd, the only real root is $+1$. Dividing out the factor $x-1$, we obtain the reciprocal equation $x^{n-1}+x^{n-2}+\ldots\ldots+x+1=0$, considered by Gauss in his *Disquisitiones Arithmeticæ*, by Bachmann in his *Kreistheilung*, and others.

My method of treatment is based upon pure geometric principles. The unknowns in my treatment are all real; in Gauss', all imaginary. Further, the number of the unknowns by my method is just half as great as the number by Gauss'.

It is necessary to consider here only regular polygons of an odd number of sides, those of an even number of sides being derived from them by continued bisection of the angles at the centre.

I will now consider the method here employed to obtain the equations whose solution will give the inscription of the regular polygons. I will first treat a couple of special cases and then pass to the general one.

To form the equation upon which depends the inscription of the regular polygon of 7 sides.

Let $7a=\pi$. Then $\sin 3a=\sin 4a$.

$$\therefore \sin 2a \cos a+\cos 2a \sin a=2 \sin 2a \cos 2a.$$

$$2 \sin a \cos^2 a+\cos 2a \sin a=4 \sin a \cos a \cos 2a.$$

Dividing out $\sin a$, $2 \cos^2 a+\cos 2a=4 \cos a \cos 2a$.

$$\therefore 2 \cos^2 a+2 \cos^2 a-1=4 \cos a(2 \cos^2 a-1)$$

$$\text{or,}\quad 8 \cos^3 a-4 \cos^2 a-4 \cos a+1=0.$$

Let $x=2 \cos a$, then $x^3-x^2-2x+1=0\ldots.(1)$.

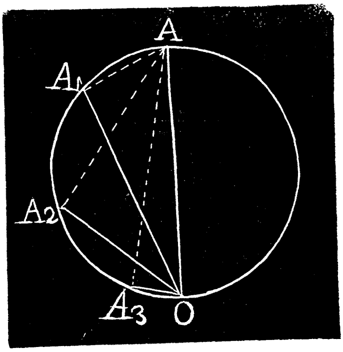
Beginning anew, let $7a'=2\pi$.

Then $\sin 3a' = -\sin 4a'$. Expanding as above, $8\cos^3 a' + 4\cos^2 a' - 4\cos a' - 1 = 0$, which may be written $(-2\cos a')^3 - (-2\cos a')^2 - 2(-2\cos a') + 1 = 0$.

Thus $-2\cos a'$ is a root of equation (1).

Finally, let $7a'' = 3\pi$. Then $\sin 3a'' = \sin 4a''$.

Hence, $8\cos^3 a'' - 4\cos^2 a'' - 4\cos a'' + 1 = 0$. Thus $2\cos a''$ is the third root of (1). These roots may evidently be written thus: $2\cos\frac{\pi}{7}$, $-2\cos\frac{2\pi}{7}$, $2\cos\frac{3\pi}{7}$.



Suppose a circle of unit radius divided at A, A_1, A_2, A_3, etc., into 7 equal parts, and the diameter AO drawn. Join the points A and O to A_1, A_2, A_3.

Then $\angle AOA_1 = \frac{\pi}{7}$; $\angle AOA_2 = \frac{2\pi}{7}$; $\angle AOA_3 = \frac{3\pi}{7}$

$\therefore 2\cos\frac{\pi}{7} = 2\cos AOA_1 = OA_1$; $2\cos\frac{2\pi}{7} = OA_2$; $2\cos\frac{3\pi}{7} = AO_3$.

Hence, $OA_1, -OA_2$, and OA_3 are the roots of the cubic

$$x^3 - x^2 - 2x + 1 = 0.$$

For a regular polygon of 9 sides, let $9a = \pi$. Then $\sin 4a = \sin 5a$. Expanding, dividing out $\sin a$, and reducing, we find $16\cos^4 a - 8\cos^3 a - 12\cos^2 a + 4\cos a + 1 = 0$. Write $x = 2\cos a$, then $x^4 - x^3 - 3x^2 + 2x + 1 = 0 \ldots\ldots (2)$.

Let $9a' = 2\pi$; then $\sin 4a' = -\sin 5a'$. Expanding we find $16\cos^4 a' + 8\cos^3 a' - 12\cos^2 a' - 4\cos a' + 1 = 0$, which may be written $(-2\cos a')^4 - (-2\cos a')^3 - 3(-2\cos a')^2 + 2(-2\cos a') + 1 = 0$. Hence, $-2\cos a'$ is a root of (2).

Let $9a'' = 3\pi$; then $\sin 4a'' = \sin 5a''$. Lastly, let $9a''' = 4\pi$; then $\sin 4a''' = -\sin 5a'''$. Hence, $2\cos a''$ and $-2\cos a'''$ are roots of equation (2). These four roots may be written $2\cos\frac{\pi}{9}, -2\cos\frac{2\pi}{9}, 2\cos\frac{3\pi}{9}, -2\cos\frac{4\pi}{9}$. These roots are easily shown to be the chords $OA_1, -OA_2, OA_3$, and $-OA_4$ in a circle of unit radius.

Since $2\cos\frac{3\pi}{9} = 1$, one root of (2) is 1, and dividing out the factor $x-1$ we obtain the cubic $x^3 - 3x - 1 = 0$, whose roots are the chords $OA_1, -OA_2, -OA_4$ in the unit circle.

Proceeding in a similar manner, we obtain the equations upon whose solution depends the inscription of other regular polygons:

For 11 sides, $x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1 = 0$, For 13 sides, $x^6-x^5-5x^4+4x^3+6x^2-3x-1=0$.

If we form a table of the coefficients of these equations, we discover a curious resemblance to the famous Triangle of Pascal, each column in the triangle gives two colums in our table. The table up to 19 sides is as follows, N being the number of sides of the regular polygon:

N										
3	1	1								
5	1	1	1							
7	1	1	2	1						
9	1	1	3	2	1					
11	1	1	4	3	3	1				
13	1	1	5	4	6	3	1			
15	1	1	6	5	10	6	4	1		
17	1	1	7	6	15	10	10	4	1	
19	1	1	8	7	21	15	20	10	5	1

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NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

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[Continued from the August Number.]

PROPOSITION VI. *The hypothesis of an obtuse angle, if even in a single case it is true, always in every case it alone is true.*

PROOF. Let the join CD (Fig. 5) make obtuse angles with any two equal perpendiculars AC, BD, standing upon any other straight AB.

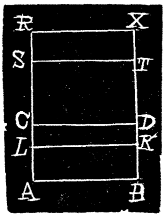

FIG. 5

CD will be (P. III.) less than this AB.

Assume in AC and BD produced any two mutually equal portions CR and DX; and join RX.

Now I investigate the angles at the join RX, which certainly (P. I.) will be mutually equal.

If they are obtuse we have our assertion.

But they are not right, because thus we would have a case for the hypothesis of right angle, which (P. V.) would have no place for the hypothesis of obtuse angle. But neither are they acute

For thus RX would be (P. III.) greater than this AB; and still more therefore greater than CD itself. But that this cannot be is thus shown. If the quadrilateral $CDXR$ is taken to be filled up by straights cutting off from